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Title of Presentation:

Asymptotically Periodic Solutions To Linear First Order ODEs

Proposed Graduation Date: 2006
Post-Graduation Plans: PhD in Physics and Computer Science

Research Experience:
David Nichols: Information theory and dynamical systems. Received Eaton grant.
Aaron Arvey: Research into linear and abstract algebra applications in factoring theory.

The general solution of the differential equation

$$x'(t)=a(t)x(t)+b(t) \qquad (1)$$

has the following form:

$$x(t)=\exp(\int_0^t a(s)ds)\left(\int_0^t b(s)\exp(-\int_0^s a(r)dr)ds+x_0\right). \qquad (2)$$

x_0 is an arbitrary initial condition at t=0.

To make this equation more transparent, we write

$$A(t)=\int_0^t a(r)dr \qquad (3)$$

$$B(t)=\int_0^t b(s)\exp(-\int_0^s a(r)dr)ds \quad (4)$$

In the new notation, the general solution becomes:

$$x(t)=e^{A(t)}B(t)+e^{A(t)}x_0 \qquad (5)$$

We first assume that $a(t)$ and $b(t)$ are continuous and periodic of period T. We want to prove the following result.

Theorem: Assume that $a(t)$ has strictly negative average, i.e.

$$\frac{1}{T}\int_0^T a(t)dt = q < 0. \qquad (6)$$

Then, the following two statements hold.

i) The differential equation (1) has one and only one strictly periodic solution $x_p(t)$ of period T.

ii) Every other solution $x(t)$ converges to $x_p(t)$, i.e.

$$\lim_{t \to +\infty} x(t) - x_p(t) = 0. \qquad (7)$$

PROOF

i) Equation (1) has one and only one periodic solution of period T.

Write $m(t)=a(t)-q$. Then $\int_0^T m(t)dt=0$ and $A(t)$ takes the form

$$A(t)=qt+\int_0^t m(s)ds \quad (8)$$

Evaluate (5) when $t=T$ and obtain

$$x(T)=\exp(qT)B(T)+x_0\exp(qT) \qquad (9)$$

To obtain a periodic solution of period T we set $x(T)=x_0$ to find

$$x_0\left(1-\exp(qT)\right)=\exp(qT)B(T) \qquad (10)$$

Since $qT\neq 0$, equation (10) provides a unique initial condition such that the corresponding solution $x_p(t)$ is periodic of period T.

ii) Every other solution converges to $x_p(t)$.

Starting with an initial condition, x_1, different from that provided by (10), we take the difference and find:

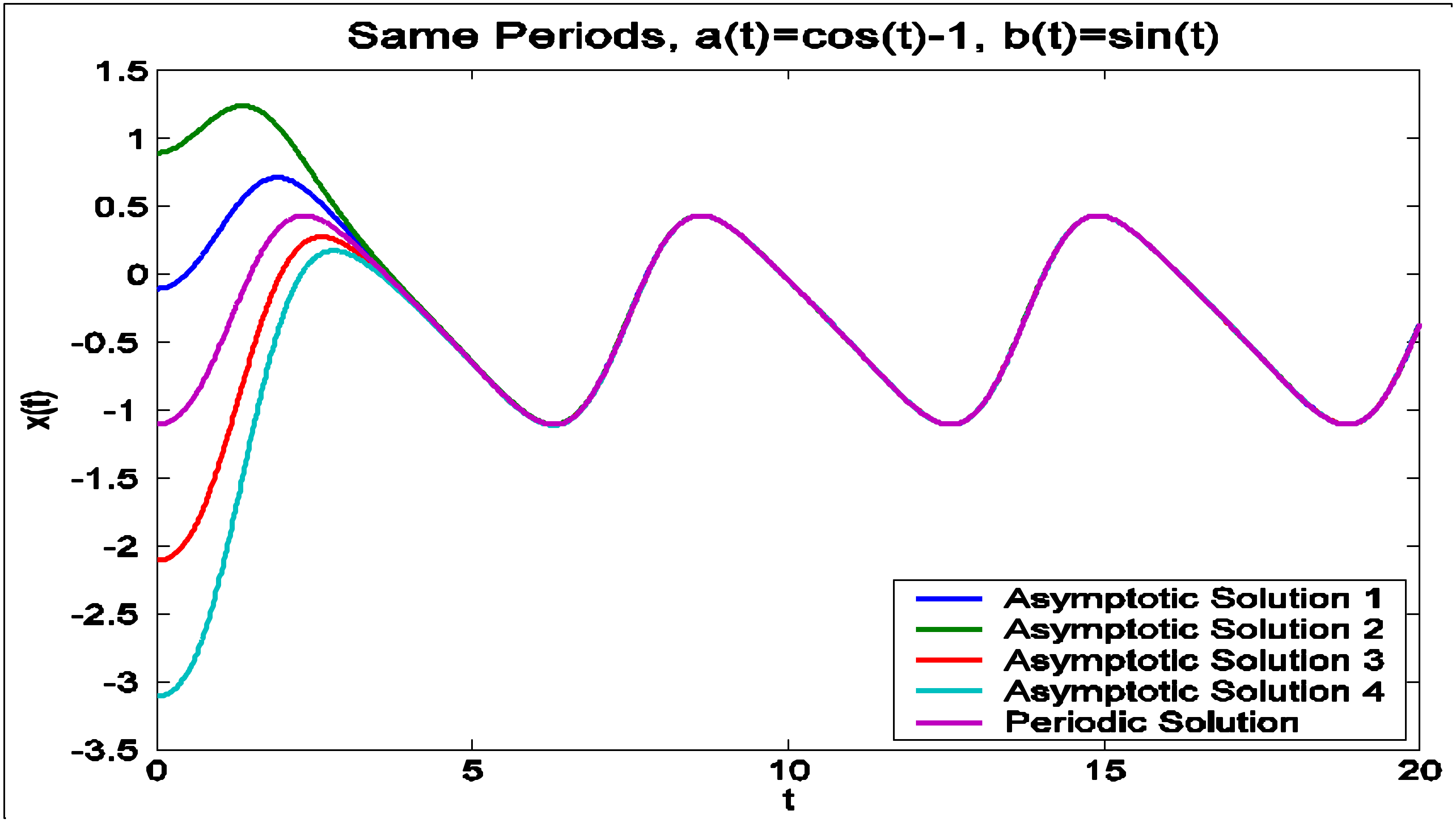
$$x(t) - x_p(t) = \exp\left(qt + \int_0^t m(s)\,ds\right)(x_1 - x_0) \qquad (11)$$

Since $m(t)$ is periodic with average 0, the term $\int_0^t m(s)ds$ is bounded. The term qt goes to $-\infty$. Thus all solutions approach $x_p(t)$.

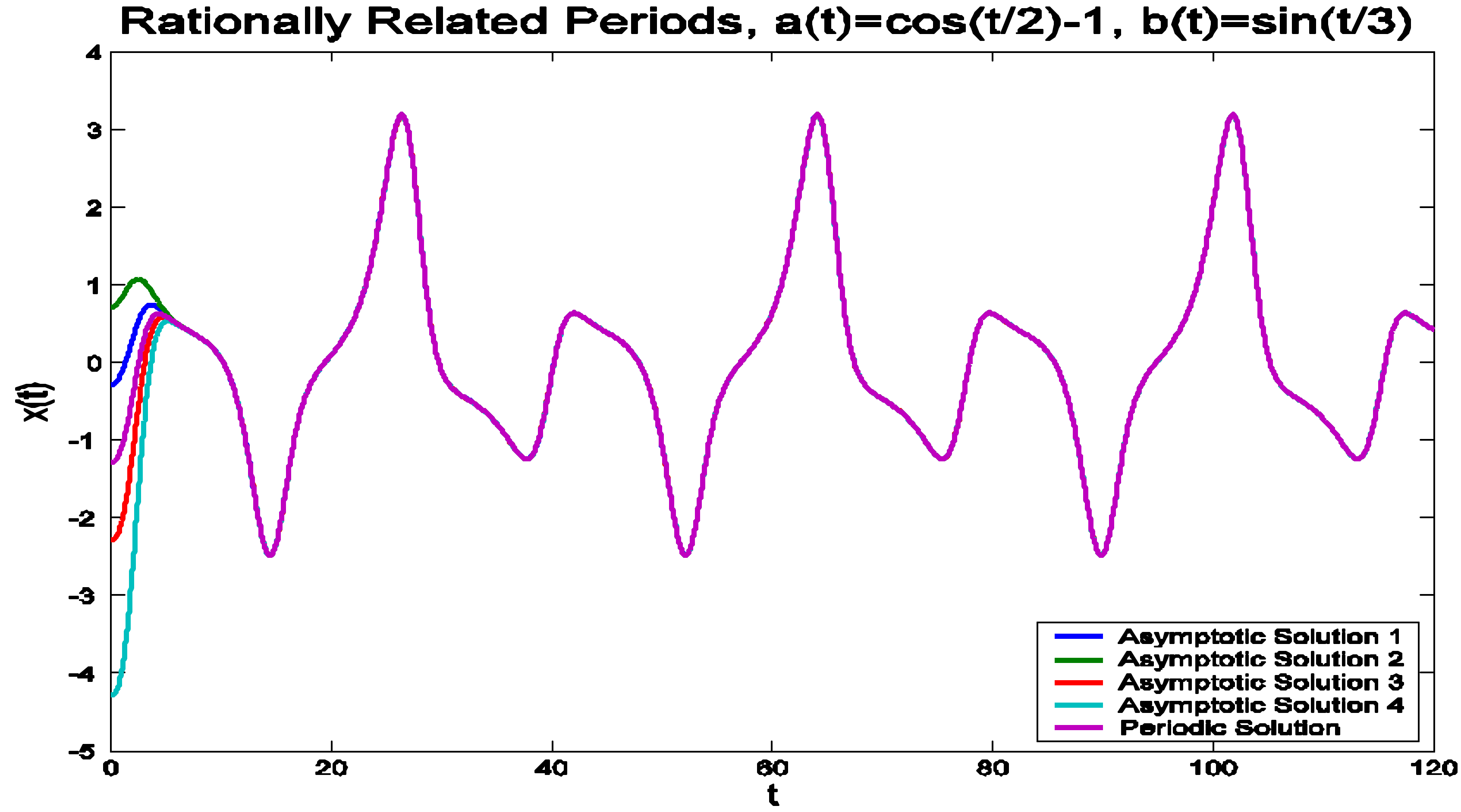
Remark:

The theorem still holds when the ratio between the periods of $a(t)$ and $b(t)$ is rational; however, the same is not true for irrationally related periods.

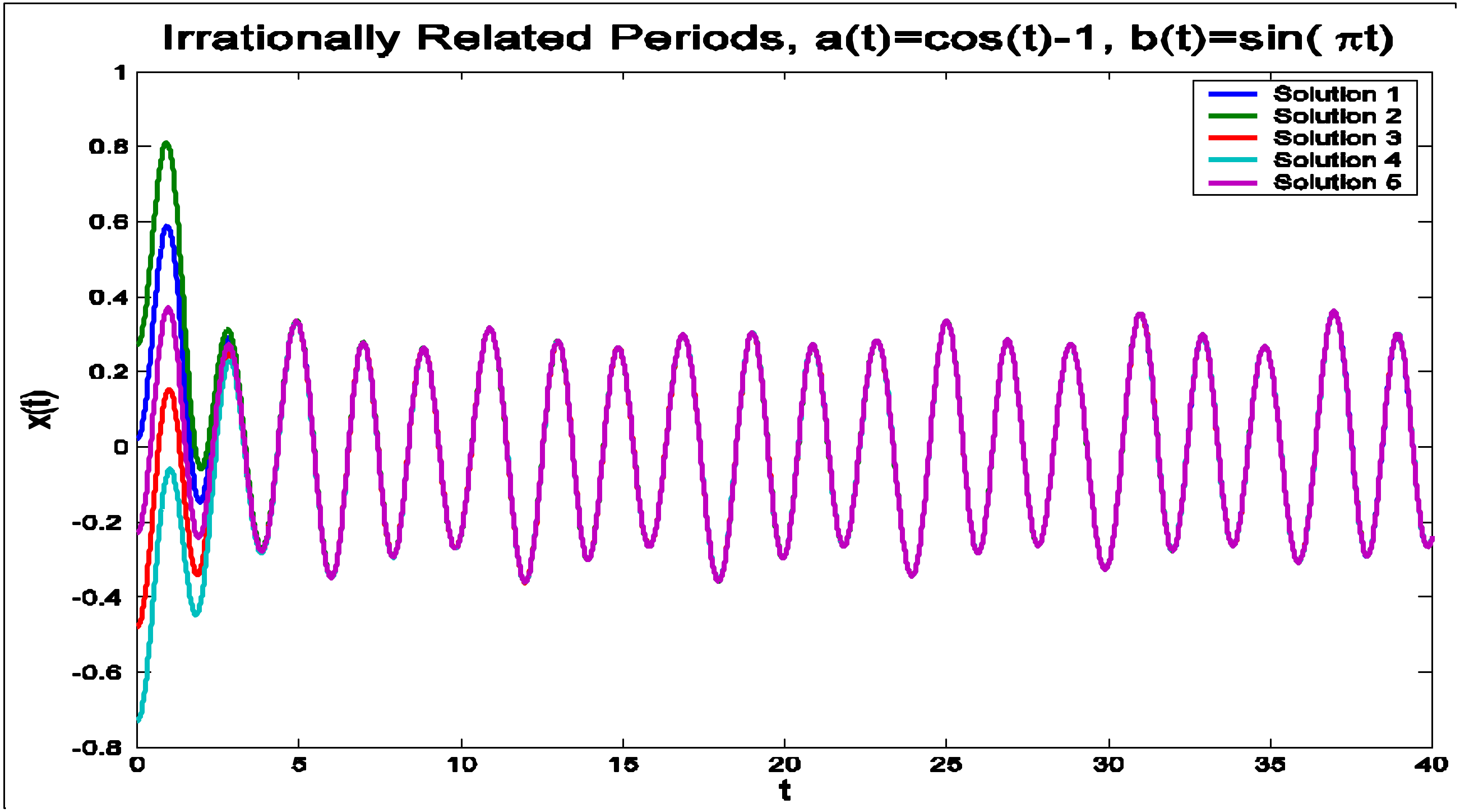
Here is the graph of a numerical simulation using the simple case, $a(t)=\cos(t)-1$ and $b(t)=\sin(t)$.



Below is an example when the ratio between the two periods is rational



When the ratio of the two periods is not rational, the first part of the theorem fails but a variant of the second holds



The result can be extended to systems. For example, consider the system in vector form:

$$\overrightarrow{x(t)}' = A(t)\overrightarrow{x(t)} + \overrightarrow{b(t)} \qquad (12)$$

and assume that the matrix $A(t)$ and the vector $\overrightarrow{b(t)}$ are periodic of the same period T. Define $A_0 = \int_0^T A(t)dt$.

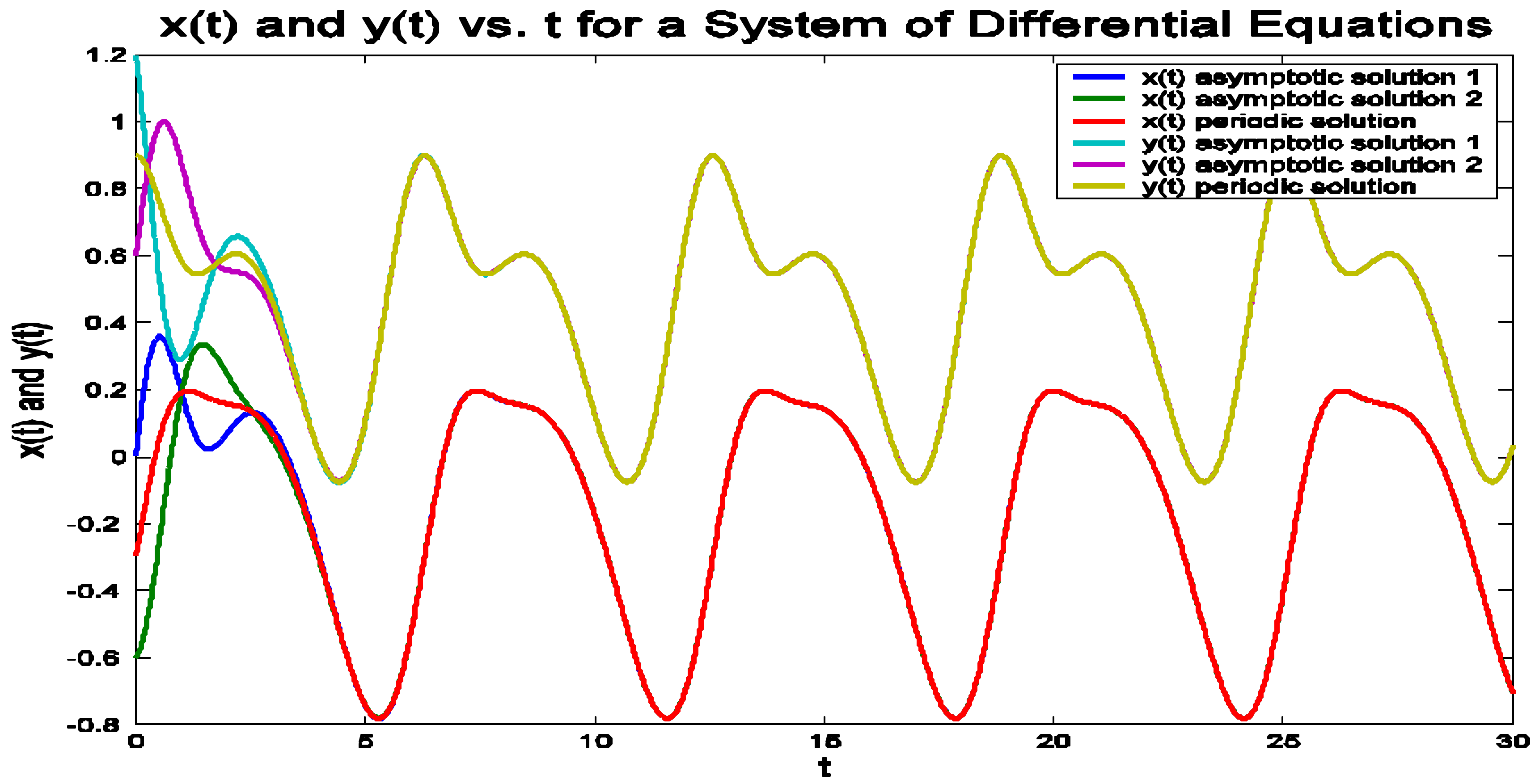
Then, the assumption

“ $a(t)$ has negative average”

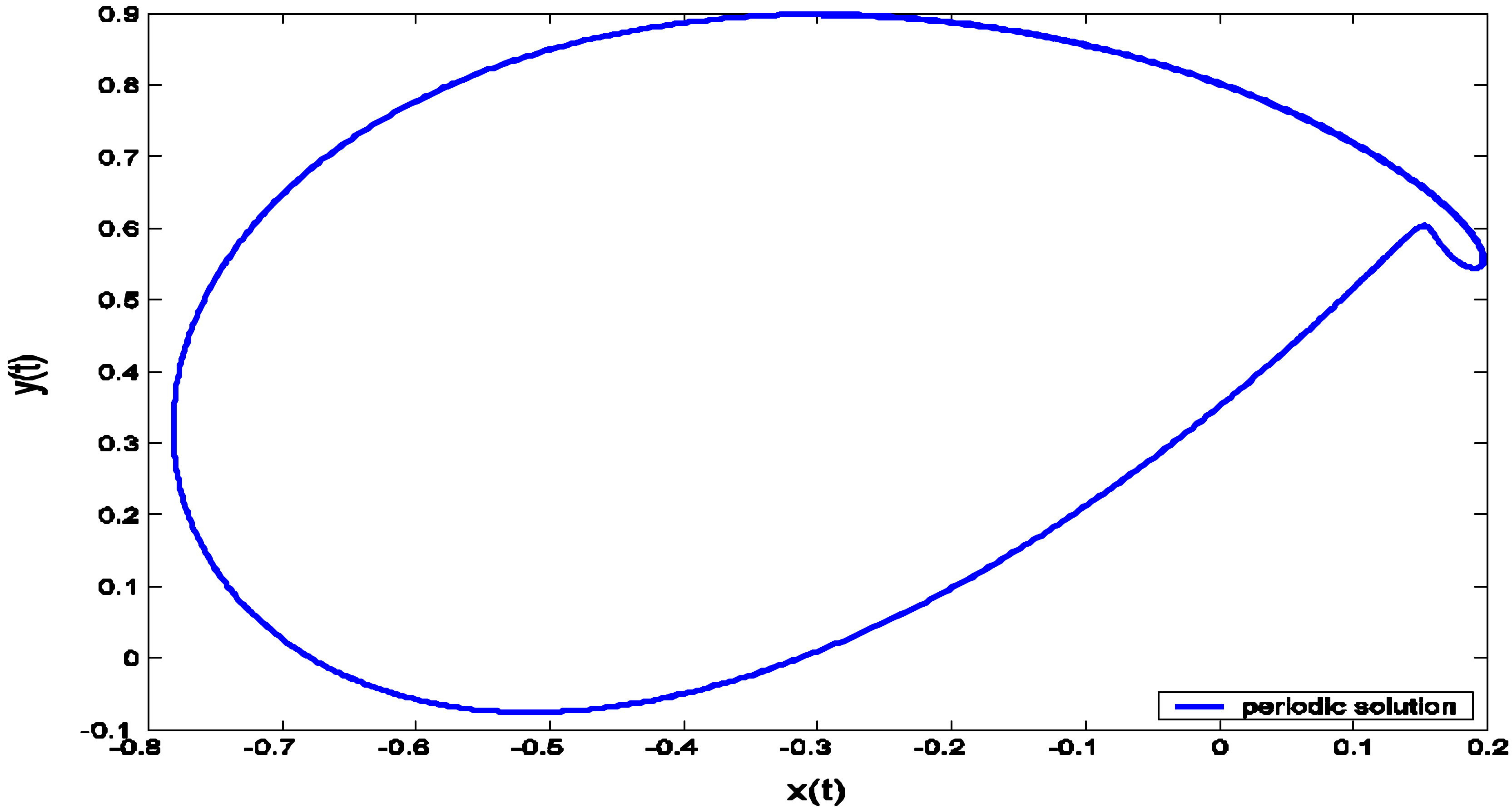
becomes

“the eigenvalues of A_0 have negative real part”.

The following example uses the matrix $A_0 = \begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix}$



Below is the phase plane trajectory of the periodic solution



THANK YOU FOR YOUR ATTENTION

END OF PRESENTATION

In order to pursue the system proof, we start by showing that given an eigenvector $\vec{v}$ and an eigenvalue l (i.e. $l\vec{v}=\vec{A}\vec{v}$) then we can say that

$$(\exp(At)-\exp(l\,It))\vec{u}=k\vec{v} \qquad (13)$$

Proof:

The proof lies in the Taylor Series expansion.